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Quantization of Curvature for Compact Surfaces in S^n

Udo SIMON
Haizhong LI

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Quantization of Curvature for Compact Surfaces in S^n *

Udo Simon, Haizhong Li

Abstract: For minimal surfaces in spheres there is a well known conjecture about the quantization of curvature which has been solved only in special cases so far. In one of such special cases we weaken the minimality assumption.

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1. Introduction

In [3], Calabi considered minimal immersions of compact surfaces with constant Gauss curvature K into $S^n(1)$. He gave a complete list of all such immersions and proved that the set of possible values of K is discrete, namely $K = K(s)$:

$$K(s) = \frac{2}{s(s+1)}, \quad s \in N.$$

This led to the so called (see [12])

Quantization Conjecture: *Let (M, g) be a compact surface minimally immersed into $S^n(1)$; denote by K the curvature of the Riemannian metric g. If*

$$K(s+1) \le K \le K(s)$$

for an $s \in N$, then either $K = K(s)$ or $K = K(s+1)$, and the immersion is one of Calabi's standard immersions.

So far, this conjecture has been solved only in the cases $s = 1$ and $s = 2$ (see [6], [1], [5]); under additional assumptions there are many partial solutions (see e.g. [10], [9], [2], [11]).

In this paper, we drop the assumption on the minimality of the immersion and consider surfaces with variable mean curvature and Gauss curvature K in the interval

$$\frac{1}{3} \le K \le 1.$$

Recall that, for a minimal immersion as considered above, $K = K(s) = 1$ for $s = 1$ gives an equator in $S^3(1)$, while $K = K(s) = 1/3$ for $s = 2$ gives a Veronese surface in $S^4(1)$ which can be described as in Example 1 below.

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We use the following notation. Let $x : M \to S^n(1)$ be a surface in an n-dimensional unit sphere $S^n(1)$. Let e_α $(3 \leq \alpha \leq n)$ are local orthonormal normal vector fields of M in $S^n(1)$. If h^α_{ij} denotes the second fundamental form with respect to e_α, $S = |II|^2$ denotes the square of the length of the second fundamental form, $\mathbf{H}$ denotes the mean curvature vector and H denotes the mean curvature of M, then we have

$$|II|^2 = S = \sum_\alpha \sum_{i,j} (h^\alpha_{ij})^2, \quad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{2} \sum_k h^\alpha_{kk}, \quad H = |\mathbf{H}|.$$

Example 1 (see [4] or [6]). Veronese surface. Let (x, y, z) denote the canonical coordinate system in R^3 and $u = (u_1, u_2, u_3, u_4, u_5)$ the canonical coordinates in R^5. We consider the mapping defined by

$$u_1 = \frac{1}{\sqrt{3}} yz, \quad u_2 = \frac{1}{\sqrt{3}} xz, \quad u_3 = \frac{1}{\sqrt{3}} xy,$$

$$u_4 = \frac{1}{2\sqrt{3}}(x^2 - y^2), \quad u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2),$$

where $x^2 + y^2 + z^2 = 3$. This defines an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$. Two points (x, y, z) and $(-x, -y, -z)$ of $S^2(\sqrt{3})$ are mapped into the same point of $S^4(1)$. This real projective plane imbedded in $S^4(1)$ is called the *Veronese surface*. We know that the Veronese surface is a minimal surface in $S^4(1)$ (see [6]). We also note that $|II|^2$ and the Gauss curvature K of the Veronese surface satisfy

$$|II|^2 = \frac{4}{3}, \qquad K = \frac{1}{3}. \tag{1.1}$$

For $s = 1$ and $n = 4$, the first result concerning the quantization of curvature was proved by B. Lawson [6]; for $s = 1$ and arbitrary n, the quantization result is the consequence of the following integral inequality for minimal immersions.

Theorem 1 (Benko-Kothe-Semmler-Simon [1] or Kozlowski-Simon [5]) *Let M be a compact minimal surface with Gauss curvature K in an n-dimensional unit sphere $S^n(1)$. Then we have*

$$\int_M (1 - K)(3K - 1) dv \leq 0. \tag{1.2}$$

In particular, if

$$\frac{1}{3} \leq K \leq 1, \tag{1.3}$$

then either $K = 1$ and M is totally geodesic, or $K = \frac{1}{3}$, $n = 4$ and M is the Veronese surface given by Example 1.

As already stated, we drop the minimality assumption and extend this quantization result as follows

Proposition *Let M be a compact surface in an n-dimensional unit sphere $S^n(1)$. If the Gauss curvature K and the mean curvature satisfy*

$$\frac{1}{3} \leq K \leq 1, \tag{1.4}$$

and

$$|\mathrm{grad} \vec{H}|^2 \leq 4H^2(2K - 1), \tag{1.5}$$

then either $K = \frac{1}{3}, n = 4$ and M is the Veronese surface given by Example 1, or $K = 1$ and the components of the second fundamental form of M are given by

$$(h_{ij}^3) = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad (h_{ij}^4) = \begin{pmatrix} 0 & \sqrt{1/2}H \\ \sqrt{1/2}H & 0 \end{pmatrix}, \quad (h_{ij}^5) = \begin{pmatrix} \sqrt{1/2}H & 0 \\ 0 & -\sqrt{1/2}H \end{pmatrix},$$

and

$$h_{ij}^\beta = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta \geq 6,$$

where H is the mean curvature function and e_3 is the unit vector of $\vec{H}$.

Quantization Theorem. *Let M be a compact surface in an n-dimensional unit sphere $S^n(1)$. If the Gauss curvature K and the mean curvature satisfy (1.4) and (1.5), then*

(i) either $K = \frac{1}{3}$, $n = 4$ and M is the Veronese surface given by Example 1,

(ii) or $K = 1$ and $|\mathrm{grad}\vec{H}|^2 = 4H^2(2K-1)$; under additional assumptions, we obtain the following classification results:

(ii.1) If H =constant, then either $n = 3$, $H = 0$ and M is totally geodesic; or $n = 5$, $H = \sqrt{2}$ and M is a Veronese surface in $S^4(\frac{1}{\sqrt{3}})$ in $S^5(1)$;

(ii.2) if $n \leq 4$, then M is totally geodesic in $S^3(1)$.

Remark 1.1. (i) In the case of minimal surfaces, the Quantization Theorem reduces to Theorem 1.

(ii) We would like to comment on the character of the condition (1.5). For hypersurfaces in Euclidean space it is known that the mean curvature is the only genuine extrinsic curvature invariant (see e.g. the introduction of [13]), while the other elementary symmetric curvature functions $H_r (r \geq 2)$ of the principal curvatures can be described (for odd order modulo sign) in terms of the metric. An analogous result is true for hypersurfaces in space forms of curvature $\bar{K}$: the curvature function $H_r (r \geq 2)$ can be described in terms of the metric and $\bar{K}$ (for odd order modulo sign); the proof follows the lines of that of Theorem 5.3 in [8]. This fact explains the particular interest in the mean curvature and in relations between H as extrinsic and K as intrinsic curvature. For higher codimension, the mean curvature and $|II|^2$ are the most important extrinsic curvature invariants. Our inequality (1.5) relates intrinsic and extrinsic curvature. While it is trivially satisfied for minimal surfaces, for $H \neq 0$ this inequality describes a quantitative control of the extrinsic mean curvature in terms of the intrinsic Gauss curvature K.

(iii) In particular for minimal submanifolds, the invariant $S = |II|^2$ is used to describe the extrinsic curvature behaviour, and for minimal hypersurfaces in a sphere, there is another well known quantization conjecture about the behaviour of S (see [4]). The class of Willmore surfaces (we will recall the definition below) enlarges that of minimal surfaces in spheres. The following result is an analogue of our foregoing quantization result for surfaces which are not necessarily minimal; it was proved in [7]; here we give a new and shorter proof.

Theorem 2 (see Li [7]). *Let M be a compact Willmore surface in an n-dimensional unit sphere $S^n(1)$. Then we have*

$$\int_M \rho^2(2-\frac{3}{2}\rho^2)dv \le 0, \tag{1.6}$$

where $\rho^2 := |II|^2 - 2H^2$. In particular, if

$$0 \le \rho^2 \le \frac{4}{3}, \tag{1.7}$$

then either $\rho^2 = 0$ and M is totally umbilic in $S^3(1)$, or $\rho^2 = \frac{4}{3}$, $n = 4$ and M is the Veronese surface given by Example 1.

Remark 1.2 If the codimension equals one, then $\rho^2 = |II|^2 - 2H^2 = \frac{1}{2}(k_1 - k_2)^2$ with k_1, k_2 as principal curvatures. The foregoing Theorem 2 is another generalization of Theorem 1 to Willmore surfaces. We can reformulate (1.6) as inequality for H and K only, without $\|\text{grad}\vec{H}\|^2$; in this form the close relation to Theorem 1 is obvious.

Corollary 1. *Let M be a compact Willmore surface in $S^n(1)$. Then*

$$\int_M \{(H^2-K)+1\}\{3(H^2-K)-1\}dv \le 0. \tag{1.8}$$

In particular, if

$$\frac{1}{3} \le K - H^2 \le 1, \tag{1.9}$$

then either $K - H^2 = 1$ and M is totally umbilic in $S^3(1)$, or $H = 0$, $K = \frac{1}{3}$ and M is the Veronese surface in $S^4(1)$.

2. Preliminaries

Let $x : M \to S^n(1)$ be a surface in an n-dimensional unit sphere. We choose an orthonormal basis $e_1, \cdots, e_n$ of $S^n(1)$ such that $\{e_1, e_2\}$ are tangent to $x(M)$ and $\{e_3, \cdots, e_n\}$ is a local frame in the normal bundle. Let $\{\omega_1, \omega_2\}$ be the dual forms of $\{e_1, e_2\}$. We use the following convention on the ranges of indices:

$$1 \le i, j, k, \cdots \le 2; \qquad 3 \le \alpha, \beta, \gamma, \cdots \le n.$$

Then we have the structure equations

$$dx = \sum_i \omega_i e_i, \tag{2.1}$$

$$de_i = \sum_j \omega_{ij} e_j + \sum_{\alpha,j} h^\alpha_{ij}\omega_j e_\alpha - \omega_i x, \tag{2.2}$$

$$de_\alpha = -\sum_{i,j} h^\alpha_{ij}\omega_j e_i + \sum_\beta \omega_{\alpha\beta}e_\beta, \quad h^\alpha_{ij} = h^\alpha_{ji}. \tag{2.3}$$

The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.4}$$

this implies

$$R_{ik} = \delta_{ik} + 2\sum_{\alpha} H^{\alpha} h^{\alpha}_{ik} - \sum_{\alpha,j} h^{\alpha}_{ij} h^{\alpha}_{jk}, \tag{2.5}$$

$$2K = 2 + 4H^2 - |II|^2; \tag{2.6}$$

as above, K is the Gauss curvature of M and $S = |II|^2 = \sum_{\alpha,i,j}(h^{\alpha}_{ij})^2$ is the square of the norm of the second fundamental form.

We consider the canonical embedding of $S^n(1)$ in R^{n+1} and describe $x(M)$ as surface in R^{n+1} in terms of its position vector, again denoted by x. R^{n+1} is equipped with an Euclidean structure, defined by the canonical scalar product $<,>$: $R^{n+1} \times R^{n+1} \to R$. Then we have

$$< x, x >= 1, \quad < x, x_i >= 0. \tag{2.9}$$

Denote covariant derivatives of x by $x_i, x_{ij}, x_{ijk}, x_{ijkl}$ and by Δ the Laplacian of the metric; then

$$x_{ij} = -\delta_{ij} x + \sum_{\alpha} h^{\alpha}_{ij} e_{\alpha}, \tag{2.10}$$

$$\Delta x = -2x + 2\vec{H}, \tag{2.11}$$

$$< \Delta x, \Delta x >= 4 + 4H^2, \quad H^2 := |\vec{H}|^2, \tag{2.12}$$

$$\sum_{j} < x_{ij}, x_{jk} >= \delta_{ik} + \sum_{\alpha} h^{\alpha}_{ij} h^{\alpha}_{jk}, \tag{2.13}$$

$$\sum_{i,j} < x_{ij}, x_{ij} >= 2 + |II|^2. \tag{2.14}$$

The vector valued Ricci identities for x read

$$x_{ijk} - x_{ikj} = \sum_{l} x_l R_{lijk}, \tag{2.15}$$

$$x_{ijkl} - x_{ijlk} = \sum_{m} x_{mj} R_{mikl} + \sum_{m} x_{im} R_{mjkl}, \tag{2.16}$$

where x_i, x_{ij}, x_{ijk} and x_{ijkl} satisfy

$$dx = \sum_{i} x_i \omega_i, \tag{2.17}$$

$$\sum_{j} x_{ij} \omega_j = dx_i + \sum_{j} x_j \omega_{ji}, \tag{2.18}$$

$$\sum_{k} x_{ijk} \omega_k = dx_{ij} + \sum_{k} x_{kj} \omega_{ki} + \sum_{k} x_{ik} \omega_{kj}, \tag{2.19}$$

$$\sum_{l} x_{ijkl} \omega_l = dx_{ijk} + \sum_{l} x_{ljk} \omega_{li} + \sum_{l} x_{ilk} \omega_{lj} + \sum_{l} x_{ijl} \omega_{lk}. \tag{2.20}$$

We have the following formula

$$\begin{aligned}\sum_k x_{ijkk} &= (\Delta x)_{ij} + \sum_m x_{mj} R_{mi} + \sum_m x_{im} R_{mj} + \sum_m x_m (R_{mi})_j \\ &\quad + \sum_{m,k} x_{mk} R_{mijk} + \sum_{l,k} x_{lk} R_{lijk} + \sum_{l,k} x_l (R_{lijk})_k .\end{aligned} \tag{2.21}$$

By use of $< x_i, x_{jk} >= 0$, we have (c.f. [5])

$$\begin{aligned} & \tfrac{1}{2}\Delta \sum_{i,j} < x_{ij}, x_{ij} > \\ = & \sum_{i,j,k} < x_{ijk}, x_{ijk} > + \sum_{i,j} < x_{ij}, (\Delta x)_{ij} > + 2K \sum_{i,j} < x_{ij}, x_{ij} > \\ & +2 \sum_{i,j,k,m} < x_{ij}, x_{mk} R_{mijk} > \\ = & \sum_{i,j,k} < x_{ijk}, x_{ijk} > + [4K(|II|^2 - 2H^2) - 4 - 2|II|^2] + 2 \sum_{i,j} < x_{ij}, \vec{H}_{ij} > . \end{aligned} \tag{2.22}$$

We extend a construction procedure from [1] where we applied it to eigenfunctions of the Laplacian.

Lemma 2.1 *We construct a totally symmetric, trace-free vector valued tensor field by*

$$W_{ijk} := x_{ijk} + \frac{1+K}{2}\delta_{ij} x_k + \frac{1-K}{2}(\delta_{ik} x_j + \delta_{jk} x_i) - \frac{1}{2}(\vec{H}_k \delta_{ij} + \vec{H}_i \delta_{jk} + \vec{H}_j \delta_{ik}), \tag{2.23}$$

we have

$$\begin{aligned}\sum < W_{ijk}, W_{ijk} > = \sum < x_{ijk}, x_{ijk} > -3(1-K)^2 - 3|\mathrm{grad}\vec{H}|^2 \\ +(K+1)(K-3) + 2(3-K)\sum < \vec{H}_k, x_k > .\end{aligned} \tag{2.24}$$

Proof. Straightforward.

Lemma 2.2 *Let $x : M \to S^n(1)$ be a compact surface, then we have*

$$2\int_m \sum < x_{ij}, \vec{H}_{ij} > = -4\int_M |\mathrm{grad}\vec{H}|^2 + 4\int_M H^2(K-2). \tag{2.25}$$

Proof. We have the following calculation

$$\begin{aligned} 2\textstyle\int_m \sum < x_{ij}, \vec{H}_{ij} > &= -2\textstyle\int_M \sum < x_{ijj}, \vec{H}_i > \\ &= -2\textstyle\int_M \sum < (\Delta x)_i + K x_i, \vec{H}_i > \\ &= -2\textstyle\int_M \sum < -2x_i + 2\vec{H}_i + K x_i, \vec{H}_i > \\ &= -4\textstyle\int_M |\mathrm{grad}\vec{H}|^2 - 2\textstyle\int_M \sum < (K-2)x_i, \vec{H}_i > \\ &= -4\textstyle\int_M |\mathrm{grad}\vec{H}|^2 + 2\textstyle\int_M \sum < [((K-2)x_i)_i, \vec{H} > \\ &= -4\textstyle\int_M |\mathrm{grad}\vec{H}|^2 + 2\textstyle\int_M \sum < (K-2)\Delta x, \vec{H} > \\ &= -4\textstyle\int_M |\mathrm{grad}\vec{H}|^2 + 4\textstyle\int_M H^2(K-2). \end{aligned}$$

3. Proofs of the Proposition and the Quantization Theorem

Proof of the Proposition Integrating (2.22), by use of (2.23) and (2.24) we have

$$\begin{aligned} 0 &= \int_M \sum < W_{ijk}, W_{ijk} > - \int_M |\mathrm{grad}\vec{H}|^2 \\ &\quad + \int_M [-2(3K-1)(K-1) + 4H^2(2K-1)]. \end{aligned} \tag{3.1}$$

In particular, if

$$\frac{1}{3} \leq K \leq 1, \tag{3.2}$$

and

$$|\mathrm{grad}\vec{H}|^2 \leq 4H^2(2K-1), \tag{3.3}$$

then we have from (3.1) either $K = 1$, or $K = \frac{1}{3}$. In these cases, we have from (3.1)

$$-|\mathrm{grad}\vec{H}|^2 + 4H^2(2K-1) = 0. \tag{3.4}$$

If $K = \frac{1}{3}$, from (3.4) we have $H = 0$ and $W_{ijk} \equiv 0$. Calabi's list of standard immersions gives $n = 4$, and M is the Veronese surface described in Example 1.

To discuss the case $K = 1$, we use the following lemma.

Lemma 3.1 *We have the following formulas*

$$x_{ijk} = -\delta_{ij}x_k - \sum h^\alpha_{ij} h^\alpha_{kl} x_l + \sum h^\alpha_{ij,k} e_\alpha, \tag{3.5}$$

and

$$\vec{H}_i = \sum H^\alpha_{,i} e_\alpha - \sum H^\alpha h^\alpha_{im} x_m, \tag{3.6}$$

where $h^\alpha_{ij,k}$ *and* $H^\alpha_{,i}$ *are defined by*

$$\sum h^\alpha_{ij,k}\omega_k = dh^\alpha_{ij} + \sum h^\alpha_{kj}\omega_{ki} + \sum h^\alpha_{ik}\omega_{kj} + \sum h^\beta_{ij}\omega_{\beta\alpha}, \tag{3.7}$$

$$\sum_i H^\alpha_{,i}\omega_i = dH^\alpha + \sum_\beta H^\beta \omega_{\beta\alpha}. \tag{3.8}$$

Proof. From $d\vec{H} = \sum \vec{H}_i\omega_i$, (3.8) and

$$d\vec{H} = d(\sum H^\alpha e_\alpha) = \sum dH^\alpha e_\alpha + \sum H^\alpha de_\alpha = \sum H^\alpha_{,i}\omega_i e_\alpha - \sum H^\alpha h^\alpha_{im}\omega_i e_m$$

we get (3.6). Putting (2.10) into (2.19), we get (3.5).

We proceed with the proof of the Proposition. When $K = 1$, the vector valued equation $0 = W_{ijk} = x_{ijk} + \delta_{ij}x_k - \frac{1}{2}(\vec{H}_k\delta_{ij} + \vec{H}_i\delta_{jk} + \vec{H}_j\delta_{ik})$ is equivalent to

$$h^\alpha_{ij,k} = \frac{1}{2}(H^\alpha_{,k}\delta_{ij} + H^\alpha_{,i}\delta_{jk} + H^\alpha_{,j}\delta_{ik}), \tag{3.9}$$

and

$$\sum_\alpha h^\alpha_{ij} h^\alpha_{mk} = \frac{1}{2}\sum H^\alpha (h^\alpha_{km}\delta_{ij} + h^\alpha_{im}\delta_{jk} + h^\alpha_{jm}\delta_{ik}). \tag{3.10}$$

Contraction of (3.10) gives

$$\sum H^\alpha h^\alpha_{ij} = |\vec{H}|^2\delta_{ij}, \tag{3.11}$$

we conclude that M is pseudo-umbilical. Choosing $e_3||\vec{H}$, we have

$$h^3_{ij} = H\delta_{ij}, \quad H^3 = H, \quad H^\beta = 0, \quad \beta \geq 4. \tag{3.12}$$

By use of (3.12), we have from (3.10)

$$\sum_\alpha (h^\alpha_{11})^2 = \sum_\alpha (h^\alpha_{22})^2 = \frac{3}{2}H^2, \quad \sum_\alpha (h^\alpha_{12})^2 = \frac{1}{2}H^2, \tag{3.13}$$

$$\sum_\alpha h^\alpha_{11}h^\alpha_{12} = \sum_\alpha h^\alpha_{22}h^\alpha_{12} = 0, \quad \sum_\alpha h^\alpha_{11}h^\alpha_{22} = \frac{1}{2}H^2. \tag{3.14}$$

Thus we can choose

$$e_3||\vec{H}, \qquad e_4||\sum_\alpha h^\alpha_{12}e_\alpha, \qquad e_5||\sum_\alpha (h^\alpha_{11} - h^\alpha_{22})e_\alpha. \tag{3.15}$$

It is easy to check that then the components of the second fundamental form satisfy

$$(h^3_{ij}) = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad (h^4_{ij}) = \begin{pmatrix} 0 & \sqrt{1/2}H \\ \sqrt{1/2}H & 0 \end{pmatrix}, \tag{3.16}$$

and

$$(h^5_{ij}) = \begin{pmatrix} \sqrt{1/2}H & 0 \\ 0 & -\sqrt{1/2}H \end{pmatrix}, \quad (h^\beta_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta \geq 6. \tag{3.17}$$

This completes the proof of the proposition.

Proof of the Quantization Theorem. From the Proposition, we only need to prove the statements $(ii.1)$ and $(ii.2)$.

By use of (3.16), we have

$$\begin{aligned} \tfrac{1}{2}\Delta(\sum_{i,j}(h^3_{ij})^2) &= \sum(h^3_{ij,k})^2 + \sum h^3_{ij}h^3_{ij,kk} \\ &= \sum(h^3_{ij,k})^2 + \sum_{i,k} Hh^3_{ii,kk}. \end{aligned} \tag{3.18}$$

If $H \neq 0$, we have the following formula (see Yau [16])

$$\sum_i h^3_{ii,kl} = 2H_{kl} - \frac{1}{2H}\sum_{\alpha\neq 3}(\sum_i h^\alpha_{ii,k})(\sum_i h^\alpha_{ii,l}), \tag{3.19}$$

which implies

$$H\sum_{i,k} h^3_{ii,kk} = 2H\Delta H - \frac{1}{2}\sum_{k,\alpha\neq 3}(\sum_i h^\alpha_{ii,k})(\sum_i h^\alpha_{ii,k}). \tag{3.20}$$

Putting (3.20) into (3.18), by use of (3.9) and (3.16) we get

$$\begin{aligned} 0 &= \textstyle\int_M \sum(h^3_{ij,k})^2 + 2\int_M H\Delta H - \int_M 2\sum_{\alpha\neq 3,i}(H^\alpha_{,i})^2 \\ &= \textstyle\int_M[|\nabla H|^2 - 2\sum_{\alpha\neq 3,i}(H^\alpha_{,i})^2]. \end{aligned} \tag{3.21}$$

$(ii.1)$ When H =constant, from (3.21) we get

$$H^\alpha_{,i} = 0, \quad 1 \leq i \leq 2, \quad 3 \leq \alpha \leq n. \tag{3.22}$$

Thus $\vec{H}$ is parallel in the normal bundle (see Yau [16]), and we have $H = 0$ or $H = \sqrt{2}$ from (3.4) and (3.6).

If $K = 1$ and $H = 0$, we know that $n = 3$ and M is totally geodesic; if $K = 1$ and $H = \sqrt{2}$, M is a Veronese surface in $S^4(\frac{1}{\sqrt{3}})$ in $S^5(1)$ from Yau's classification result of surfaces with parallel mean curvature vector (see [16]).

$(ii.2)$ In case $n \leq 4$, from the proof of the foregoing proposition, we know $H = 0$ when $K = 1$. Thus $(ii.2)$ follows from the Proposition. This completes the proof of the Quantization Theorem.

4. Proof of Theorem 2

Define

$$\tilde{h}^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}, \tag{4.1}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}^{\alpha}_{ij}\tilde{h}^{\beta}_{ij}, \tag{4.2}$$

$$\rho^2 = \sum_{i,j} (\tilde{h}^{\alpha}_{ij})^2 = |II|^2 - 2H^2. \tag{4.3}$$

By use of (2.6) and (4.3), (3.1) can be rewritten as

$$\begin{aligned} 0 &= \textstyle\int_M \sum < W_{ijk}, W_{ijk} > - \int_M |\text{grad}\vec{H}|^2 \\ &\quad + \textstyle\int_M [\rho^2(2 - \frac{3}{2}\rho^2) + 2H^2\rho^2 + 2H^4]. \end{aligned} \tag{4.4}$$

From (3.6) and (4.2), we have

$$|\text{grad}\vec{H}|^2 = \sum_{\alpha,i} (H^{\alpha}_{,i})^2 + \sum H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta} + 2H^4. \tag{4.5}$$

We recall the following definition of Willmore surfaces

Definition (see [15] or [7]) *Let M be a surface in $S^n(1)$, M is called a Willmore surface if it satisfies*

$$\Delta^{\perp}H^{\alpha} + \sum_{\beta,i,j} h^{\alpha}_{ij}h^{\beta}_{ij}H^{\beta} - 2H^2H^{\alpha} = 0, \quad 3 \leq \alpha \leq n, \tag{4.6)}$$

where $\Delta^{\perp}$ is the Laplacian in the normal bundle of M.

The following lemma can be found in Li [7]

Lemma 4.1 (see Lemma 2.6 of [7]) *Let $x : M \to S^n(1)$ be a compact Willmore surface in an n-dimensional unit sphere $S^n(1)$. Then*

$$\int_M \sum_{\alpha,i} (H^{\alpha}_{,i})^2 = \int_M \sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta}. \tag{4.7}$$

Thus, for a compact Willmore surface $x : M \to S^n(1)$, from (4.4), (4.5) and (4.7) we have

$$\begin{aligned} 0 &= \textstyle\int_M \sum < W_{ijk}, W_{ijk} > + \int_M \rho^2(2 - \frac{3}{2}\rho^2) \\ &\quad + \textstyle\int_M 2[H^2\rho^2 - \sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta}] \\ &\geq \textstyle\int_M \rho^2(2 - \frac{3}{2}\rho^2), \end{aligned} \tag{4.8}$$

where we used

$$H^2\rho^2 = (\sum_\alpha (H^\alpha)^2)(\sum_\beta \tilde\sigma_{\beta\beta}) \geq \sum_{\alpha,\beta} H^\alpha H^\beta \tilde\sigma_{\alpha\beta}. \tag{4.9}$$

In particular, if

$$0 \leq \rho^2 \leq \frac{4}{3}, \tag{4.10}$$

then either $\rho^2 = 0$ and M is totally umbilic, or $\rho^2 = \frac{4}{3}$. In the latter case, we have from (4.7)

$$W_{ijk} \equiv 0, \quad H^2\rho^2 = \sum_{\alpha,\beta} H^\alpha H^\beta \tilde\sigma_{\alpha\beta}. \tag{4.11}$$

By use of (3.5), (3.6) and (2.23), we directly check that the tangent part W^T_{ijk} of W_{ijk} is given by

$$\begin{aligned} W^T_{ijk} &= -\delta_{ij}x_k - \sum h^\alpha_{ij}h^\alpha_{kl}x_l + \tfrac{1+K}{2}\delta_{ij}x_k \\ &\quad + \tfrac{1-K}{2}(\delta_{ik}x_j + \delta_{jk}x_i) + \tfrac{1}{2}\sum(h^\alpha_{km}\delta_{ij} + h^\alpha_{im}\delta_{jk} + h^\alpha_{jm}\delta_{ik})H^\alpha x_m. \end{aligned} \tag{4.12}$$

(4.12) implies

$$\begin{aligned} \sum_\alpha h^\alpha_{ij}h^\alpha_{kl} &= -\delta_{ij}\delta_{kl} + \tfrac{1+K}{2}\delta_{ij}\delta_{kl} + \tfrac{1-K}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) \\ &\quad + \tfrac{1}{2}(h^\alpha_{kl}\delta_{ij} + h^\alpha_{il}\delta_{jk} + h^\alpha_{jl}\delta_{ik})H^\alpha. \end{aligned} \tag{4.13}$$

Contraction of (4.13) gives

$$\sum_\alpha H^\alpha h^\alpha_{ij} = H^2\delta_{ij}. \tag{4.14}$$

From (4.14), we get

$$\sum_{\alpha,\beta} H^\alpha H^\beta \tilde\sigma_{\alpha\beta} = \sum_{\alpha,\beta} H^\alpha H^\beta (h^\alpha_{ij}h^\beta_{ij} - 2H^\alpha H^\beta) = 0. \tag{4.15}$$

(4.11) and (4.15) imply $H = 0$, thus $n = 4$, and M is the Veronese surface given by Example 1.

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Udo Simon and H. Li
Institut für Mathematik, MA 8-3
Technische Universität Berlin
Strasse des 17. Juni 136, 10623 Berlin
Germany
E-mail:simon@math.tu-berlin.de
hli@math.tu-berlin.de

H. Li
Department of Mathematical Sciences
Tsinghua University
100084, Beijing
People's Republic of China
E-mail: hli@math.tsinghua.edu.cn